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Dispersion by random velocity fields

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A simple approximation is proposed for the problem of the dispersion of marked particles in an incompressible fluid in random motion when the probability distribution of the velocity field is taken as Gaussian, homogeneous, isotropic, stationary and of zero mean. Approximations for the Lagrangian velocity correlation function and the dispersion are given and compared with exact computer calculations of Kraichnan. Agreement is found to be good except for time-independent velocity fields and singular wavenumber spectral functions.

1. Introduction

An important feature of turbulent motion is the enhancement of diffusion processes to which it gives rise. A complete discussion of this would necessitate a satisfactory theory of turbulence, which does not yet exist, and it is therefore of interest to examine the simpler problem in which the fluid velocity field has a prescribed probability distribution. As well as giving useful information about the diffusion occurring in turbulent motion this problem is also of interest since it contains the same basic closure difficulty as turbulence theory without the additional complexity of nonlinear dynamics. It can therefore serve as a test case for approximation procedures whose aim is the resolution of the closure difficulty. It should be mentioned too that there are many closely related problems in other fields; wave propagation in random media and the quantum mechanics of a particle moving in a disordered lattice are just two which come to mind.

For mathematical simplicity it seems natural to consider first the case in which the Eulerian velocity field $\mathbf{u}(\mathbf{x}, t)$ has a Gaussian probability distribution. To simplify matters further we shall assume that the distribution is homogeneous, isotropic, stationary and of zero mean, and that the velocity field satisfies the incompressibility condition. As is well known, an exactly Gaussian velocity field lacks certain important features which are present in actual turbulent flow, namely nonlinear energy transfer and the passive convection of small spatial scales of motion by larger ones. However, experimental evidence seems to show that the statistics of homogeneous turbulence are close to Gaussian except for the smaller scales of motion (Frenkiel & Klebanoff 1967*a*, *b*). Hence the Gaussianity assumption should give reasonable approximations for quantities which are dominated by contributions from the large scales, such as the Langrangian velocity correlation function and the dispersion for time values which are not too small.

The problem may be formulated in either Eulerian or Lagrangian form. The Eulerian formulation is obtained by writing down an equation for the quantity

$$\phi(\mathbf{x}, t) = \delta\{\mathbf{x} - \mathbf{X}(t)\}, \tag{1}$$

where $\mathbf{X}(t)$ is the position of a particular fluid particle at time t. The equation in question is of course the continuity equation

$$(\partial/\partial t + u_\alpha(\mathbf{x}, t)\, \partial/\partial x_\alpha)\, \phi(\mathbf{x}, t) = 0. \tag{2}$$

The probability density of the position $\mathbf{x}$ of the particle at time t is then given by the expectation value $\langle \phi(\mathbf{x}, t)\rangle$.

This approach has formed the basis of most previous work on the problem. Stochastic-model theory (Kraichnan 1961) applied to (2) yields the direct-interaction approximation, which gives a nonlinear integral equation for the probability density. The same approximation may also be obtained by a self-consistent expansion procedure similar to one formulated by the author (1969) for the full turbulence problem. The Wiener–Hermite expansion method has also been applied to the problem by Saffman (1969). These approximations become asymptotically exact when the correlation time of the velocity field is very much less than the eddy circulation time (defined as the correlation length over the root-mean-square velocity). They also satisfy realizability conditions since they are derived from approximations for the random quantity ϕ itself. Neither method however ensures the positivity of the probability density.

Computer simulations of particle diffusion have been carried out by Kraichnan (1970*a*). In these a Gaussian velocity field was represented by a linear superposition of a hundred Fourier modes with random amplitudes and phases. The equation for the particle motion was solved numerically and expectation values of quantities of interest were calculated by averaging over a large number of realizations. Good agreement was found (in three dimensions) between the direct-interaction approximation and these 'exact' results for several quantities of interest, including the Lagrangian velocity correlation function

$$u_L(t) = \tfrac{1}{3}\langle \dot{\mathbf{X}}(0) . \dot{\mathbf{X}}(t)\rangle$$

and the dispersion $Y(t) = \frac{1}{3}\langle [\mathbf{X}(t)]^2\rangle$.

The Lagrangian formulation is provided by the random equation

$$\mathbf{X}(t) = \int_0^t d\tau\, \mathbf{u}(\mathbf{X}(\tau), \tau), \tag{3}$$

where, without loss of generality, we have assumed the particle to start from the origin at time zero. A solution of (3) in the form of a power series in t may easily be written down and from it power series for $u_L(t)$ and $Y(t)$ may be obtained. Even if such series were convergent, which seems unlikely, they are clearly useless for large t. An interesting attempt has been made by Kraichnan (1970*b*) to extract useful information from such power series. This is achieved by replacing the power series with another expansion by a procedure which involves the expression of the Fourier transform in terms of a suitable set of orthogonal polynomials. For reasons which are not yet clear, truncation of the resulting series gives good approximations.

Several successive-approximation procedures are known for non-random equations, such as the Newton–Kantorovich method (see, for example, Krasnosel'skii *et al.* 1972, chap. 3). Some of these procedures have been investigated for random equations with a view to obtaining general existence theorems and convergence proofs (Bharucha-Reid 1964) but no concrete applications to particular problems appear to have been made. We shall examine here the feasibility of the simplest of these successive-approximation methods for the problem in hand. This will be done simply by comparing the values of the Lagrangian velocity correlation function and the dispersion obtained from the approximation with the exact values of Kraichnan referred to above. No rigorous discussion of convergence properties will be attempted.

The simplest approximation technique generates from an initial approximation $\mathbf{X}^{(0)}(t)$ an infinite chain of approximants given by the formula

$$\mathbf{X}^{(n+1)}(t) = \int_0^t d\tau\, \mathbf{u}(\mathbf{X}^{(n)}(\tau), \tau).$$

The nth approximant for $u_L(t)$ is then $\frac{1}{3}\langle \dot{\mathbf{X}}^{(n)}(0) . \dot{\mathbf{X}}^{(n)}(t)\rangle$.

An extremely crude assessment of the convergence properties of the sequence can be ascertained by writing

$$\begin{aligned}\langle[\mathbf{X}^{(n+1)}(t) - \mathbf{X}^{(n)}(t)]^2\rangle \\ \approx \left\langle \int_0^t d\tau \int_0^t d\tau' [X_\beta^{(n)}(\tau) - X_\beta^{(n-1)}(\tau)]\,[X_\gamma^{(n)}(\tau') - X_\gamma^{(n-1)}(\tau')]\right. \\ \left. \times\, u_{\alpha,\beta}(\mathbf{X}^{(n-1)}(\tau), \tau)\, u_{\alpha,\gamma}(\mathbf{X}^{(n-1)}(\tau'), \tau') \right\rangle,\end{aligned}$$

where the comma notation denotes spatial differentiation. The right-hand side is of order $V^2TtL^{-2}\langle[\mathbf{X}^{(n)}(t) - \mathbf{X}^{(n-1)}(t)]^2\rangle$ for $t > T$, where V is the root-mean-square velocity, L the correlation length and T the correlation time of the velocity field. One therefore expects the sequence to converge for values of t/T appreciably less than $(L/VT)^2$. If the ratio L/VT is large compared with unity then the sequence should converge rapidly for all t values of interest. Unfortunately this argument tells us nothing about the case of real turbulence, for which $L/VT \approx 1$.

A fact which soon becomes apparent is that, in general, useful closed expressions are provided by the method only for $n \leqslant 2$. However, as will be demonstrated below, the second approximant is quite good except for frozen fields and singular wavenumber spectral functions. For the frozen-field case the third approximant can be expressed in the form of a multiple integral but this has not yet been calculated.

2. The iteration procedure and approximations

The initial approximation will be taken as zero, i.e. $\mathbf{X}^{(0)}(t) \equiv 0$. The first and second iterates are then given by

$$\mathbf{X}^{(1)}(t) = \int_0^t d\tau\, \mathbf{u}(0, \tau),$$

$$\mathbf{X}^{(2)}(t) = \int_0^t d\tau\, \mathbf{u}\left(\int_0^\tau d\tau'\, \mathbf{u}(0, \tau'), \tau\right).$$

Hence we have

$$\langle \dot{\mathbf{X}}^{(2)}(0).\dot{\mathbf{X}}^{(2)}(t)\rangle = \left\langle \mathbf{u}(0,0).\mathbf{u}\left(\int_0^t d\tau\,\mathbf{u}(0,\tau),t\right)\right\rangle.$$

Introducing the delta function, this can be written as

$$\int d\mathbf{x}\left\langle \mathbf{u}(0,0).\mathbf{u}(\mathbf{x},t)\,\delta\left\{\mathbf{x}-\int_0^t d\tau\,\mathbf{u}(0,\tau)\right\}\right\rangle$$

and using the Fourier representation of the delta function it becomes

$$\frac{1}{(2\pi)^3}\int d\mathbf{x}\int d\mathbf{k}\,e^{+i\mathbf{k}.\mathbf{x}}\left\langle \mathbf{u}(0,0).\mathbf{u}(\mathbf{x},t)\exp\left(-i\mathbf{k}.\int_0^t d\tau\,\mathbf{u}(0,\tau)\right)\right\rangle. \tag{4}$$

The expectation value in this expression can be obtained by functional differentiation of the identity

$$\left\langle \exp\left[i\int d\mathbf{y}\int d\tau\,\boldsymbol{\phi}(\mathbf{y},\tau).\mathbf{u}(\mathbf{y},\tau)\right]\right\rangle$$

$$= \exp\left[-\frac{1}{2}\int \phi_\alpha(\mathbf{y}_1,\tau_1)\,R_{\alpha\beta}(\mathbf{y}_1-\mathbf{y}_2,\tau_1-\tau_2)\,\phi_\beta(\mathbf{y}_2,\tau_2)\,d\mathbf{y}_1 d\tau_1 d\mathbf{y}_2 d\tau_2\right]$$

followed by the substitution

$$\boldsymbol{\phi}(\mathbf{y},\tau) = -\mathbf{k}\,\delta(\mathbf{y})\,\theta(t-\tau)\,\theta(\tau),$$

where θ is the step function defined by

$$\theta(\tau) = \begin{cases} 0, & \tau < 0, \\ 1, & \tau > 0, \end{cases}$$

and R is the correlation function of the velocity field defined by

$$\langle u_\alpha(\mathbf{x},t)\,\mathbf{u}_\beta(\mathbf{x}',t')\rangle = R_{\alpha\beta}(\mathbf{x}-\mathbf{x}',t-t').$$

In order to be able to compare our results with those of Kraichnan we shall confine ourselves to correlation functions of the form

$$R_{\alpha\beta}(\mathbf{x},t) = \frac{1}{4\pi}\int d\mathbf{k}\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)\frac{E(k)}{k^2}\,e^{-i\mathbf{k}.\mathbf{x}}\,D(t),$$

with

$$D(0) = 1, \quad \int_0^\infty dk\,E(k) = \tfrac{3}{2}v_0^2.$$

Making use of the incompressibility condition, the expectation value in (4) can now be written as

$$2D(t)\int_0^\infty dk\,E(k)\exp\left\{-\tfrac{1}{2}v_0^2k^2\int_0^t d\tau_1\int_0^t d\tau_2\,D(\tau_1-\tau_2)\right\}$$

and we thus have finally

$$u_L(t) = \tfrac{2}{3}D(t)\int_0^\infty dk\,E(k)\exp\left\{-\tfrac{1}{2}v_0^2k^2\int_0^t d\tau_1\int_0^t d\tau_2 D(\tau_1-\tau_2)\right\}. \tag{5}$$

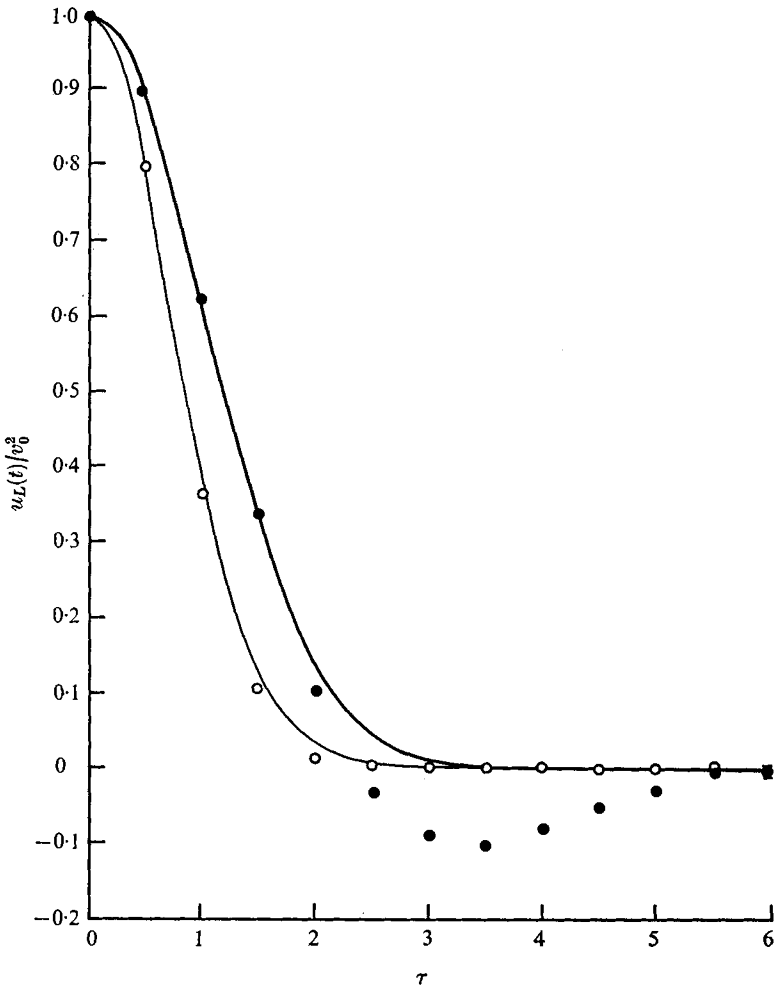

FIGURE 1. The Lagrangian velocity correlation function as a function of time: upper curve, case (i); lower curve, case (ii). Kraichnan's exact values: ●, case (i); ○, case (ii).

The four cases computed by Kraichnan have a $D(t)$ of the form

$$D(t) = \exp\left(-\tfrac{1}{2}\omega_0^2 t^2\right)$$

and are given by

(i) $E(k) = \frac{3}{2}v_0^2\delta(k-k_0), \quad \omega_0 = 0,$
(ii) $E(k) = \frac{3}{2}v_0^2\,\delta(k-k_0), \quad \omega_0 = k_0 v_0,$
(iii) $E(k) = 16(2/\pi)^{\frac{1}{2}}\,v_0^2 k_0^{-5}k^4\exp\left(-2k^2/k_0^2\right), \quad \omega_0 = 0,$
(iv) $E(k) = 16(2/\pi)^{\frac{1}{2}}\,v_0^2 k_0^{-5}k^4\exp\left(-2k^2/k_0^2\right), \quad \omega_0 = k_0 v_0.$

The approximations for $v_0^{-2}u_L(t)$ from expression (5) corresponding to these are respectively

(i) $\exp\left(-\frac{1}{2}\tau^2\right),$
(ii) $\exp\{1-\frac{1}{2}\tau^2-\exp\left(-\frac{1}{2}\tau^2\right)-\left(\frac{1}{2}\pi\right)^{\frac{1}{2}}\tau\,\mathrm{erf}\,(\tau/\sqrt{2})\},$

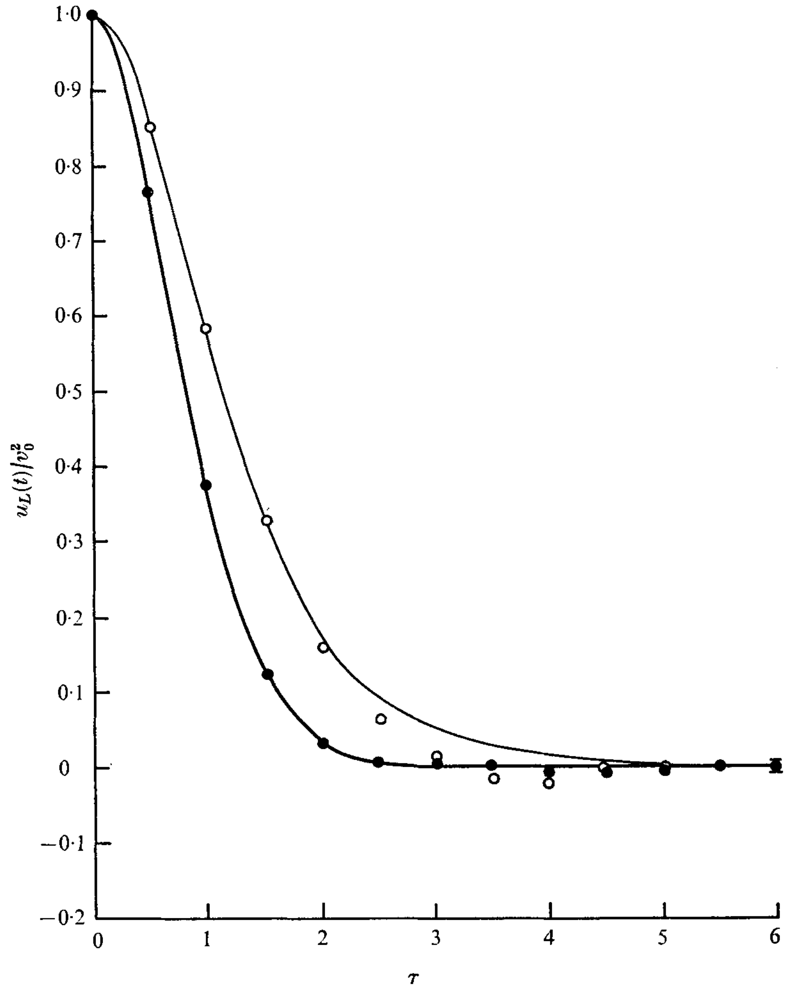

FIGURE 2. The Lagrangian velocity correlation function as a function of time: upper curve, case (iii); lower curve, case (iv). Kraichnan's exact values: ○, case (iii); ●, case (iv).

(iii) $(1+\frac{1}{4}\tau^2)^{-\frac{5}{2}}$,

(iv) $\exp(-\frac{1}{2}\tau^2)/\{1+\frac{1}{2}[(\frac{1}{2}\pi)^{\frac{1}{2}}\tau\,\mathrm{erf}(\tau/\sqrt{2})-1+\exp(-\frac{1}{2}\tau^2)]\}^{\frac{5}{2}}$,

where we have put $\tau = k_0 v_0 t$. The diagrams show these functions together with points representing the exact values.

The exact values of the dispersion and the eddy viscosity, defined as

$$\kappa(t) = \tfrac{1}{3}\langle \mathbf{X}(t)\,.\,\dot{\mathbf{X}}(t)\rangle,$$

are related to the Lagrangian velocity correlation function by the equations

$$\kappa(t) = \int_0^t d\tau\, u_L(\tau), \quad Y(t) = 2\int_0^t d\tau\, \kappa(\tau).$$

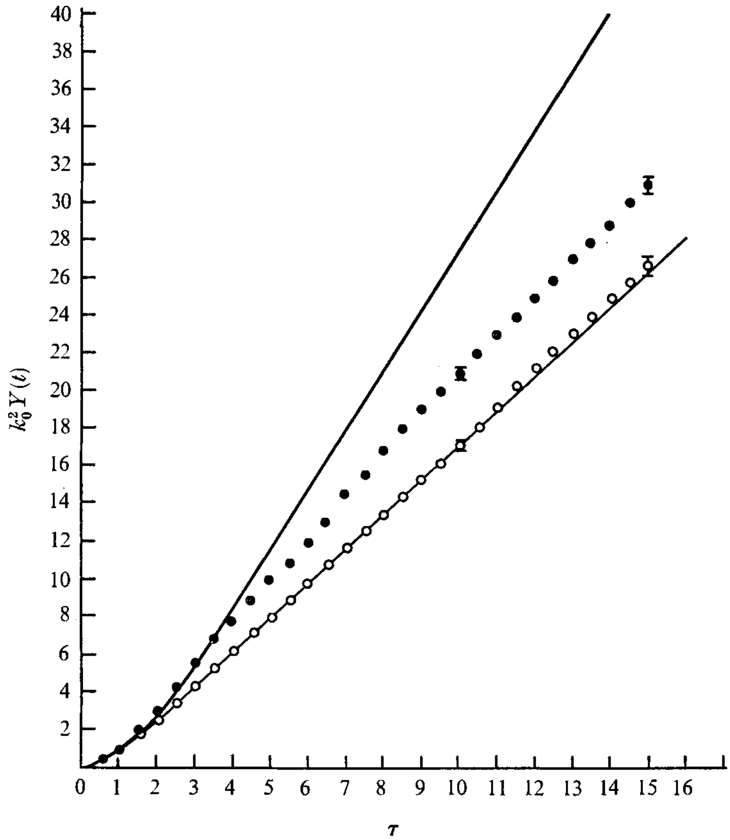

FIGURE 3. The dispersion as a function of time: upper curve, case (i); lower curve, case (ii). Kraichnan's exact values: ●, case (i); ○, case (ii).

The first of these is not necessarily true however for the approximate values; that is, we have in general

$$\langle \mathbf{X}^{(n)}(t) . \dot{\mathbf{X}}^{(n)}(t) \rangle \neq \int_0^t d\tau \langle \dot{\mathbf{X}}^{(n)}(0) . \dot{\mathbf{X}}^{(n)}(\tau) \rangle .$$

The expression for $\langle [\mathbf{X}^{(2)}(t)]^2 \rangle$ is rather complicated so, at this stage, we shall adopt the simpler expedient of taking as our approximation for the dispersion the quantity

$$\frac{2}{3} \int_0^t d\tau \int_0^\tau d\tau' \langle \dot{\mathbf{X}}^{(2)}(0) . \dot{\mathbf{X}}^{(2)}(\tau') \rangle .$$

Graphs of this are shown for the four cases together with the exact values of the dispersion.

3. Conclusion

It will be seen that the method is rather more successful than the crude convergence argument would suggest, and is particularly good for case (iv), which is more characteristic of real turbulence. In case (i) it is less impressive since it

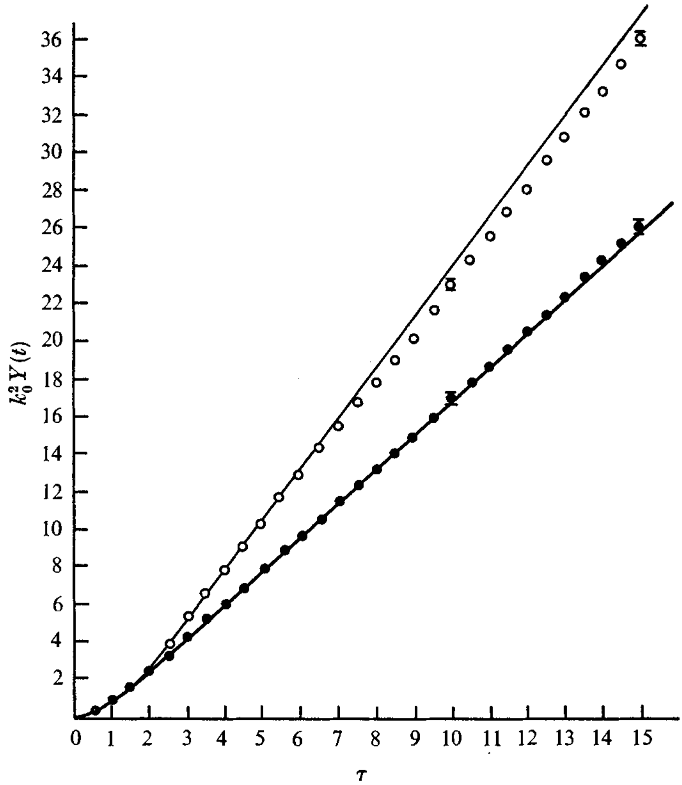

FIGURE 4. The dispersion as a function of time: upper curve, case (iii); lower curve, case (iv). Kraichnan's exact values: ○, case (iii); ●, case (iv).

fails to reproduce the negative region of $u_L(t)$ and consequently overestimates the dispersion. This negative region of $u_L(t)$ is associated with the negative values of the Eulerian correlation function, as a function of relative position, which occur for the spectral function of case (i). It should also be noted that the approximation presented is exact in the two simple cases which can be solved exactly: namely, uniform velocity fields, and velocity fields with delta-function time correlations. Our tentative conclusion is that the approximation will be reasonable for almost any spectral function if the ratio $R = L/VT$ is not much smaller than unity. For a sufficiently smooth spectral function much smaller values of R may be accommodated. It is not possible to be more precise until exact calculations for other spectral functions and R values are available for comparison.

Unfortunately the method does not give equally simple approximations for the probability density. For the frozen-field case the spatial Fourier transform can be expressed as

$$\frac{1}{(2\pi)^{\frac{3}{2}}} v_0^{-3} \int d\mathbf{v} \exp\left\{-\frac{1}{2v_0^2}\left[v^2 + 2ik_\alpha v_\beta \int_0^t d\tau R_{\alpha\beta}(\tau\mathbf{v}) + k_\alpha k_\beta \int_0^t d\tau \int_0^t d\tau' [v_0^2 R_{\alpha\beta}\{(\tau-\tau')\mathbf{v}\} - R_{\alpha\gamma}(\tau\mathbf{v}) R_{\beta\gamma}(\tau'\mathbf{v})]\right]\right\}.$$

In general, however, one obtains a functional integral. Similarly the third approximant for $u_L(t)$ can only be written in a useful closed form for the frozen-field case. We obtain

$$u_L(t) = \frac{1}{3}\frac{v_0^{-5}}{(2\pi)^{\frac{9}{2}}}\int d\mathbf{v}\int d\mathbf{k}\, v_\alpha \tilde{R}_{\alpha\beta}(\mathbf{k})\left\{v_\beta + ik_\gamma\int_0^t d\tau R_{\gamma\beta}(\tau\mathbf{v})\right\}$$
$$\times\exp\left\{-\frac{1}{2v_0^2}\left[v^2 + 2ik_\gamma v_\delta\int_0^t d\tau R_{\gamma\delta}(\tau\mathbf{v})\right.\right.$$
$$\left.\left.+k_\gamma k_\delta\int_0^t d\tau\int_0^t d\tau'[v_0^2 R_{\gamma\delta}\{(\tau-\tau')\mathbf{v}\} - R_{\gamma\epsilon}(\tau\mathbf{v})\,R_{\delta\epsilon}(\tau'\mathbf{v})]\right]\right\}.$$

These integrals could be computed numerically if the isotropy condition were used.

An interesting question is whether the method would prove equally successful if the conditions of homogeneity, isotropy and zero mean velocity were dropped. It would then certainly be necessary to consider more carefully the choice of the initial approximation. Probably the best choice would be to take the mean particle position as the initial approximation; this position would then be determined self-consistently. At second order this would lead to the equation

$$\boldsymbol{\phi}(t) = \left\langle\int_0^t d\tau\,\mathbf{u}\left(\int_0^\tau d\tau'\mathbf{u}(\boldsymbol{\phi}(\tau'),\tau'),\tau\right)\right\rangle,$$

the right-hand side of which can be rewritten, in the usual way, in terms of the correlation function and the mean of the velocity field. Finally there is the possibility of attacking the turbulence problem by using such a successive-approximation technique for the Lagrangian form of the Navier–Stokes equation. These questions are now being investigated.

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